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ROCKET BOOSTER CONTROL

A SUFFICIENT CONDITION IN OPTIMAL CONTROL

by E. B. Lee

Prepared under Contract No. NASw-563 *by*
MINNEAPOLIS-HONEYWELL REGULATOR COMPANY
Minneapolis, Minnesota
for

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ABSTRACT

A theorem is proven which covers most of the known cases where L. S. Pontriagins' Maximum Principle is a sufficient, as well as a necessary, condition for optimal control. Author

INTRODUCTION

Consider the system

$$\dot{x}^{o} = f^{o}(x,t) + h^{o}(u,t) \tag{1}$$

$$\dot{x} = A(t)x + h(u,t) \tag{2}$$

with $x(t_o) = x_o$ and $x^o(t_o) = 0$. Here f^o, h^o, A, and h are continuous in all arguments. x is the system state, an n vector; u is the control, an m vector. x^o is a scalar variable which measures the quality of control.

If $u(s)$ is any control function on the interval $[t_o,t]$ we will write the corresponding response of equations (1) and (2) as $\hat{x}_u(t) = (x_u^o(t), x_u(t))$. The control u is restricted to a set $\Omega \subset R^m$. It is assumed that either Ω is compact or that h, and h^o are such that

$$\max_{u\in\Omega}\left\{\lambda\cdot h(u,t) + \lambda^o h^o(u,t)\right\}$$

exists for each $t\in[t_o,T]$ and $\hat{\lambda} = (\lambda^o,\lambda)\in R^{n+1}$ with $\lambda^o < 0$.

It is further assumed that $f^o(x,t)$ is a single-valued, convex function of x for each $t\in[t_o,T]$, that is,

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* Prepared under contract NASw-563 for the NASA

‡ Research Consultant, Minneapolis-Honeywell Regulator Company, Minneapolis, Minnesota.

$$\frac{\partial f^o}{\partial x}(x,t)\cdot(\omega-x) + f^o(x,t) \leq f^o(\omega,t)$$

for all x and $\omega \epsilon R^n$ and $t \epsilon [t_o,T]$.

The following definitions are needed:

DEFINITION 1. $u^*(s)$ on $[t_o,T]$ is an extremal control if there exists $\lambda(t_o) = (\lambda_o^1,\lambda_o^2,\ldots,\lambda_o^n)$ and λ^o = constant < 0, such that

$$\lambda^o h^o(u^*(s),s) + \lambda(s)\cdot h(u^*(s),s) = \max_{u\epsilon\Omega}\{\lambda^o h^o(u,s) + \lambda(s)\cdot h(u,s)\}$$

with

$$\mathring{\lambda} = - A'(t)\lambda - \lambda^o \frac{\partial f^o}{\partial x}(x_{u*}(t),t), \quad (\text{' denotes transpose}).$$

Here $x_{u*}(t)$ is the response corresponding to $u^*(s)$, $t_o \leq s \leq t \leq T$.

DEFINITION 2. The control $u(s)$ is allowable if it is a measurable real valued vector function with range in Ω on $[t_o,T]$.

DEFINITION 3. The set of attainability $K(T,x_o)$ is the collection of end points of the responses $\hat{x}_u(t) = (x_u^o(t),x_u(t))$ which initiate at $(0,x_o)$ for all allowable controls $u(s)$ on $[t_o,T)$.

The problem of optimal control studied here is to select allowable controls $u(s)$ which "steer" the response $x_u(t)$ from the initial point x_o at time t_o to a prescribed target set G at time $T < \infty$ and minimize the cost functional of control $C(u) = g(x(T)) + x^o(T)$. Here g is a continuously differentiable function of x. An allowable control which provides an absolute minimum for $C(u)$ amongst the set of all allowable controls which steer $x_u(t)$ from x_o to G is called an optimal control. Note, the free end point problem results when $G = R^n$.

DEVELOPMENT

A basic inequality is established and then the main theorem is proven.

LEMMA. Let $u^*(s)$, $t_o \leq s \leq T$, be an allowable extremal control with corresponding response $\hat{x}_{u*}(t)$ which initiates at $\hat{x}(t_o)=(0,x(t_o))$, then $\hat{\lambda}(T)\cdot\hat{x}_{u*}(T) \geq \hat{\lambda}(T)\cdot\hat{\omega}$ for $\lambda^o < 0$ and all $\hat{\omega}\epsilon K(T,x_o)$.

PROOF. Consider

$$\frac{d(\hat{\lambda}\cdot\hat{x}_{u*})}{dt} = \lambda^o \dot{x}^o_{u*} + \dot{\lambda}\cdot x_{u*} + \lambda\cdot\dot{x}_{u*}$$

$$= \lambda^o(f^o(x_{u*},t) + h^o(u^*,t)) +$$

$$(-A'(t)\lambda - \lambda^o \frac{\partial f^o}{\partial x}(x_{u*},t))\cdot x_{u*}$$

$$+ \lambda\cdot(A(t)x_{u*} + h(u^*,t)).$$

Upon integrating both sides between t_o and T we obtain

$$\lambda^o x^o_{u*}(T) + \lambda(T)\cdot x_{u*}(T) - \lambda(t_o)\cdot x(t_o) =$$

$$\int_{t_o}^{T} \{\lambda^o(f^o(x_{u*}(t),t) - \frac{\partial f^o}{\partial x}(x_{u*}(t),t)\cdot x_{u*}(t))$$

$$+ \lambda^o h^o(u^*(t),t) + \lambda(t)\cdot h(u^*(t),t)\}dt$$

Let $x_u(t)$ be any other response with initial value $x_o = x(t_o)$ for which we calculate

$$\lambda^o x^o_u(T) + \lambda(t)\cdot x_u(T) - \lambda(t_o)\cdot x(t_o) =$$

$$\int_{t_o}^{T} \{\lambda^o(f^o(x_u(t),t) - \frac{\partial f^o}{\partial x}(x_{u*}(t),t)\cdot x_u(t))$$

$$+ \lambda^o h^o(u(t),t) + \lambda(t)\cdot h(u(t),t)\}dt.$$

But $\lambda^o h^o(u^*(t),t) + \lambda(t)\cdot h(u^*(t),t) \geq \lambda^o h^o(u(t),t) + \lambda(t)\cdot h(u(t),t)$.

Thus if $\lambda^o(f^o(x_{u*}(t),t) - \frac{\partial f^o}{\partial x}(x_{u*}(t),t)\cdot x_{u*}(t)) \geq \lambda^o(f^o(x_u(t),t)$

$$- \frac{\partial f^o}{\partial x}(x_{u*}(t),t)\cdot x_u(t))$$

we obtain the desired inequality. This is certainly true if $\lambda^o < 0$ and

$$\frac{\partial f^o}{\partial x}(x_{u*}(t),t)\cdot x_{u*}(t) - f^o(x_{u*}(t),t) \geq \frac{\partial f^o}{\partial x}(x_{u*}(t),t)\cdot x_u(t) - f^o(x_u(t),t),$$

which is the convexity condition on f^o.

Thus we have

$$\lambda^o x^o_{u*}(T) + \lambda(T)\cdot x_{u*}(T) \geq \lambda^o x^o_u(T) + \lambda(T)\cdot x_u(T) \text{ or}$$

$$\hat{\lambda}(T)\cdot\hat{x}_{u*}(T) \geq \hat{\lambda}(T)\cdot\hat{\omega} \text{ all } \hat{\omega}\in K(T,x_o) \text{ and the lemma}$$

is established.

The basic inequality of the lemma enables us to establish the sufficiency of the maximum principle in a number of cases. These results are summarized as a theorem:

THEOREM.

A) Consider the cost functional of control $C(u) = x^o(T)$ and as target set G a point x_1. Let $u*(s)$, $t_o \leq s \leq T$, be an allowable extremal control which steers the corresponding response $x_{u*}(t)$ from x_o at t_o to x_1 at T, then $u*(s)$ is an optimal control.

B) Consider the cost functional $C(u) = g(x(T)) + x^o(T)$ with $g(x)$ a convex function of x and consider the target set $G = R^n$, (this is the free end point problem). Let $x(t_o) = x_o$. Then $u*(s)$, $t_o \leq s \leq T$, is an optimal control if it is an allowable extremal control with $\hat{\lambda}(T) = (-1, -\frac{\partial g}{\partial x}(x_{u*}(T)))$, (The condition on $\hat{\lambda}(T)$ is called a transversality condition).

C) Consider the cost functional $C(u) = x^{o}(T)$ and the convex, closed, target set $G = \{x \mid \gamma(x) \leq c\} \subset R^{n}$, where γ is differentiable and c a constant. Let $u^*(s)$, $t_o \leq s \leq T$, be an allowable extremal control which steers $x_{u*}(t)$ from x_o at t_o to $x_1 \epsilon G$ at T with $\lambda(T)$ an interior normal‡ to G at $x_{u*}(T)$ on ∂G, then $u^*(s)$ is optimal if such a control exists. (If there is no such $u^*(s)$ then the minimum may occur interior to G in which case B) applies with $\hat{\lambda}(T) = (-1,0,0\ldots 0)$ and if G is just one point part A) is obtained.)

PROOF:

A) From the lemma

$$\hat{\lambda}(T)\cdot\hat{x}_{u*}(T) \geq \hat{\lambda}(T)\cdot\hat{\omega} \text{ for } \lambda^{o} < o \text{ all } \hat{\omega}\epsilon K(T,x_o).$$

Thus

$$\lambda(T)\cdot x_{u*}(T) + \lambda^{o}x^{o}_{u*}(T) \geq \lambda(T)\cdot x_{u}(T) + \lambda^{o}x^{o}_{u}(T).$$

But, comparing only those responses that end at x_1, that is, those for which $x_{u*}(T) = x_u(T) = x_1$, the basic inequality becomes

$$\lambda^{o}x^{o}_{u*}(T) \geq \lambda^{o}x^{o}_{u}(T).$$

Since $\lambda^{o} < 0$ we have $C(u^*) = x^{o}_{u*}(T) \leq x^{o}_{u}(T) = C(u)$ and therefore $u^*(s)$ is optimal.

B) With $\hat{\lambda}(T) = (-1, \frac{-\partial g}{\partial x}(x_{u*}(T)))$ the inequality of the lemma is

$$\frac{-\partial g}{\partial x}(x_{u*}(T))\cdot x_{u*}(T) - x^{o}_{u*}(T) \geq \frac{-\partial g}{\partial x}(x_{u*}(T))\cdot x_{u}(T) - x^{o}_{u}(T)$$

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‡ λ is an interior normal to G at x on ∂G if λ is orthogonal to a support plane of G at x and is directed into the halfspace containing G. Thus G need not have an interior to have interior normals. Note if G does not have an interior we can still approximate it by a $\gamma(x)$.

Adding and subtracting $g(x_{u*}(T))$ on the left side and $g(x_u(T))$ on the right side the last inequality becomes

$$-x^{o}_{u*}(T) - g(x_{u*}(T)) + g(x_{u*}(T)) - \frac{\partial g}{\partial x}(x_{u*}(T))\cdot x_{u*}(t) \geq$$

$$-x^{o}_{u}(T) - g(x_{u}(T)) + g(x_{u}(T)) - \frac{\partial g}{\partial x}(x_{u*}(T))\cdot x_{u}(T).$$

But,

$$\frac{\partial g}{\partial x}(x_{u*}(T))\cdot(x_{u*}(T) - x_u(T)) + g(x_u(T)) \geq g(x_{u*}(T))$$

if g is a convex function of x.

Therefore

$$-C(u^*) = -x^{o}_{u*}(T) - g(x_{u*}(T)) \geq$$
$$-x^{o}_{u}(T) - g(x_u(T)) = -C(u),$$

or $C(u^*) \leq C(u)$. Hence part B) is established.

C) Assume for simplicity that γ was picked to be a convex function on G with $\partial G = \{x|\gamma(x) = c\}$. Consider only boundary points $x_{u*}(T)$ where it is required that $\lambda(T) = k\{-\frac{\partial \gamma}{\partial x}(x_{u*}(T))\}$ in order for $\lambda(T)$ to be an interior normal to G at $x_{u*}(T)$ on ∂G, (let $k = 1$).

The inequality of the lemma can then be written

$$\lambda(T)\cdot x_{u*}(T) = \lambda^{o}x^{o}_{u*}(T) + \left\{\frac{-\partial\gamma}{\partial x}(x_{u*}(T))\right\}\cdot x_{u*}(T) \geq$$

$$\lambda(T)\cdot x_{u}(T) = \lambda^{o}x^{o}_{u}(T) + \left\{\frac{-\partial\gamma}{\partial x}(x_{u*}(T))\right\}\cdot x_{u}(T)$$

If $x_{u*}(T)$ in on the boundary of G it is also true that $\gamma(x_{u*}(T)) = c \geq \gamma(x_u(T))$ for all allowable responses $x_u(T)\epsilon G$. Adding the last two inequalities we obtain

$$\lambda^{o}x^{o}_{u*}(T) - \frac{\partial\gamma}{\partial x}(x_{u*}(T))\cdot x_{u*}(T) + \gamma(x_{u*}(T)) \geq$$
$$\lambda^{o}x^{o}_{u}(T) - \frac{\partial\gamma}{\partial x}(x_{u*}(T))\cdot x_{u}(T) + \gamma(x_{u}(T)).$$

But again if $x_{u*}(T)$ is on ∂G and $x_u(T)$ is in the convex set G we have

$$\frac{\partial\gamma}{\partial x}(x_{u*}(T))\cdot[x_{u*}(T) - x_u(T)] + \gamma(x_u(T)) \geq \gamma(x_{u*}(T))$$

and therefore

$$C(u^*) \leq C(u). \quad \text{Q.E.D.}$$

REMARKS

If the set of attainability is closed there will exist optimum control provided there is at least one control that steers the response to the desired end point $x_1 \epsilon G$, assuming G is also closed. The property of closure is discussed in reference 2 in which a bibliography and discussion of cases are presented. The set of attainability is also known to be closed if

$$h(u,t) = B(t)u, \; f^o(x,t) = x\cdot W(t)x \text{ and } h^o(u,t) = u\cdot U(t)u,$$

for $W(t)$, $U(t)$ positive definite on $[t_o,T]$.

When the set of attainability is closed, in the above case, the inequality of the lemma establishes that its lower (exterior normal with $\lambda^o < 0$) surface is convex. For if it was otherwise we would be led to a contradiction of the maximum principle. Note that the transversality condition (reference 1) follows from the established inequality of the lemma since $\hat{\lambda}(T)$ must be an exterior normal of $\overline{K}(T,x_o)$ at the corresponding response end point, $\hat{x}_{u*}(T)$.

CONCLUSIONS

It has been proven the Maximum Principle is often a sufficient as well as necessary condition for optimal control.

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